# INDIAN STATISTICAL INSTITUTE
Probability Theory I: B. Math (Hons.) I
Semester I, Academic Year 2023-24
Final Exam

Date: 17/11/2023   Full Marks: 50   Duration: 2:00 - 5:00 PM

- **Show all your work and write explanations when needed. If you are using a result stated and/or proved in class, please quote it correctly.**

- **You are NOT allowed to use class notes, books, homework solutions, list of theorems, formulas etc.**

1. Consider the following schematic diagram of a drainage network model (as described in class), where each of the five paths is open with probability $p = 0.5$, and the paths behave independently of each other.
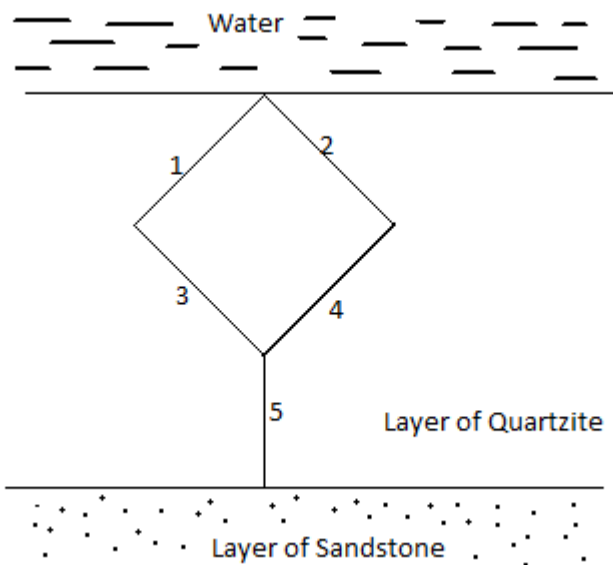
   (a) (10 marks) Let X be the number of open paths and Y be the indicator that water can pass through the layer of quartzite to the layer of sandstone. Find the joint probability mass function of X and Y.

   (b) (2+2 marks) Find the marginal probability mass functions of X and Y.

Plese Turn Over

2. (12 marks) Suppose r ($\in \mathbb{N}$) distinct toys are distributed at random among n (≥ 2) children. Let Z denote the number of children who receive exactly one toy. Calculate the expected value of Z.

3. (12 marks) Roads A and B are the only escape routes from a prison. Prison records show that of the prisoners who tried to escape, 40% used road A, and 60% used road B. The records also show that 80% of those who tried to escape via A, and 70% of those who tried to escape via B were captured. Suppose that two prisoners have independently and successfully escaped from the prison. What is the conditional probability that they used the same road to escape?

4. (12 marks) Suppose N is a nonnegative integer valued random variable that satisfies
$$\sum_{j=1}^{\infty} P(N \geq j) < \infty.$$

Then show that N has finite mean and
$$E(N) = \sum_{j=1}^{\infty} P(N \geq j).$$